Online Appendix to "Sticky Wages, Private Consumption, and Fiscal Multipliers"

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Appendix A Proofs of Theorems and Lemmas

A.1 Proof of Theorem 1

Proof. The second-order difference equation in $\hat{\pi}_t$ is given by

$$\beta E_t\left(\hat{\pi}_{t+2}\right) - \left[1+\beta+\kappa\xi - s\kappa\left(1/\alpha-1\right)\lambda\right] E_t\left(\hat{\pi}_{t+1}\right) + \hat{\pi}_t = \kappa\left(1-s\right)\left(1/\alpha-1\right)\left(1-\rho\right)\hat{g}_t$$

The two roots of the characteristic equation are

$$e_1 = \frac{\Phi + \sqrt{\Phi^2 - 4\beta}}{2\beta}$$

and

$$e_2 = \frac{\Phi - \sqrt{\Phi^2 - 4\beta}}{2\beta}$$

where $\Phi = 1+\beta+\kappa\xi - s\kappa\left(1/\alpha-1\right)\lambda$. We have the following two cases to consider.
Case (i):

$$\kappa\xi - s\kappa\left(1/\alpha-1\right)\lambda < 0 \iff \lambda > \frac{\alpha\xi}{s(1-\alpha)}$$

If $\Phi - 2\beta > 0 \iff \lambda < \frac{\alpha(1-\beta+\kappa\xi)}{s\kappa(1-\alpha)}$, then the smaller root $e_2 > \frac{[1+\beta+\kappa\xi-s\kappa(1/\alpha-1)\lambda]-\sqrt{[1-\beta+\kappa\xi-s\kappa(1/\alpha-1)\lambda]^2}}{2\beta} = 1$. Hence both roots are outside the unit circle and the equilibrium is locally unique.
If $\Phi - 2\beta < 0 \iff \lambda > \frac{\alpha(1-\beta+\kappa\xi)}{s\kappa(1-\alpha)}$, then the larger root $-1 < e_1 < 1$ and hence the equilibrium is indeterminate.

Case (ii):

$$\kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda > 0 \iff \lambda < \frac{\alpha\xi}{s(1-\alpha)}$$

The two roots are

$$\begin{aligned} e_1 &= 1 + \frac{\Phi - 2\beta + \sqrt{\Phi^2 - 4\beta}}{2\beta} \\ &= 1 + \frac{\Phi - 2\beta + \sqrt{(1-\beta)^2 + 2(1+\beta)[\kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda] + [\kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda]^2}}{2\beta} \end{aligned}$$

and

$$\begin{aligned} e_2 &= 1 + \frac{\Phi - 2\beta - \sqrt{\Phi^2 - 4\beta}}{2\beta} \\ &= 1 + \frac{\Phi - 2\beta - \sqrt{(1-\beta)^2 + 2(1+\beta)[\kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda] + [\kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda]^2}}{2\beta} \end{aligned}$$

Since

$$\begin{aligned} [1 - \beta + \kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda]^2 &= (1-\beta)^2 + 2(1-\beta)[\kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda] + [\kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda]^2 \\ &< (1-\beta)^2 + 2(1+\beta)[\kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda] + [\kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda]^2 \end{aligned}$$

we have that $e_1 > 1$ and $-1 < e_2 < 1$. Hence the equilibrium is not unique in case (ii). Overall, λ should satisfy $\frac{\alpha\xi}{s(1-\alpha)} < \lambda < \frac{\alpha(1-\beta+\kappa\xi)}{s\kappa(1-\alpha)}$ so that the equilibrium is locally unique. □

A.2 Proof of Theorem 2

Proof. First, substitute out $\hat{y}_t$ from equation (2) into (1) and get

$$\hat{\pi}_t = \kappa\{(1/\alpha - 1)\,[s\hat{c}_t + (1-s)\hat{g}_t] - \xi\hat{p}_t\} + \beta E_t\hat{\pi}_{t+1}$$

Update t to $t+1$ and take expectation.

$$E_t\hat{\pi}_{t+1} = \kappa\{(1/\alpha - 1)\,[sE_t\hat{c}_{t+1} + (1-s)E_t\hat{g}_{t+1}] - \xi E_t\hat{p}_{t+1}\} + \beta E_t\hat{\pi}_{t+2}$$

Subtract $\hat{\pi}_t$ from $E_t\hat{\pi}_{t+1}$ and then use (3) to return a second-order difference equation in $\hat{\pi}_t$.

$$\beta E_t\,(\hat{\pi}_{t+2}) - [1 + \beta + \kappa\xi - s\kappa\,(1/\alpha - 1)\,\lambda]\,E_t\,(\hat{\pi}_{t+1}) + \hat{\pi}_t = \kappa\,(1-s)\,(1/\alpha - 1)\,(1-\rho)\,\hat{g}_t$$

Next, guess a solution that takes the form $\hat{\pi}_t = \gamma \hat{g}_t$ and plug that guess into the above equation.

$$\rho^2 \beta \gamma \hat{g}_t - \rho \left(1 + \beta + \kappa \xi - s\kappa \left(1/\alpha - 1\right) \lambda\right) \gamma \hat{g}_t + \gamma \hat{g}_t = \kappa \left(1 - s\right) \left(1/\alpha - 1\right) \left(1 - \rho\right) \hat{g}_t$$

Then, the solution for the undetermined coefficient γ is

$$\gamma = \frac{\kappa \left(1/\alpha - 1\right) \left(1 - s\right) \left(1 - \rho\right)}{1 + \beta\rho^2 + \left[s\kappa \left(1/\alpha - 1\right) \lambda - \left(1 + \kappa\xi + \beta\right)\right]\rho}$$

Using the solution for inflation as a function of government purchases, we can rewrite the inflation Euler equation as

$$\left(1 - \rho\beta\right) \gamma \hat{g}_t + \kappa\xi\gamma\hat{g}_t + \kappa\xi\hat{p}_{t-1} = s\kappa \left(1/\alpha - 1\right) \hat{c}_t + \kappa \left(1/\alpha - 1\right) \left(1 - s\right) \hat{g}_t$$

Next, we rearrange this expression to have

$$\hat{c}_t = \frac{\left[1 + \kappa\xi - \rho\beta\right] \gamma - \kappa \left(1/\alpha - 1\right) \left(1 - s\right)}{s\kappa \left(1/\alpha - 1\right)} \hat{g}_t + \frac{\alpha\xi}{(1 - \alpha)s} \hat{p}_{t-1}$$

□

A.3 Proof of Lemma 1

Proof. From equation (7) and our assumption that $\rho < 1$, $\gamma > 0$ if and only if

$$1 + \beta\rho^2 > \rho \left[1 + \beta + \kappa\xi - s\kappa\lambda \left(1/\alpha - 1\right)\right]$$

Rearranging this expression,

$$\kappa^{-1}\rho^{-1} \left(1 - \rho\right) \left(1 - \beta\rho\right) > \xi - s\lambda \left(1/\alpha - 1\right)$$

The term on the left-hand side is positive given our restrictions on ρ, κ and β. The term on the right-hand side is negative because of our restriction that guarantees local uniqueness. □

A.4 Proof of Lemma 2

Proof. According to equation (8), $\chi > 0$ if and only if

$$\frac{1-\rho}{1-\rho+\frac{s\kappa(1/\alpha-1)\lambda\rho-\kappa\xi}{1-\beta\rho+\kappa\xi}} > 1$$

Recall that local uniqueness requires

$$\frac{\alpha\xi}{s\left(1-\alpha\right)} < \lambda < \frac{\alpha\left(1-\beta+\kappa\xi\right)}{s\kappa\left(1-\alpha\right)}$$

Given that $\lambda > \frac{\alpha\xi}{s(1-\alpha)}$, we have

$$1-\rho+\frac{s\kappa\left(1/\alpha-1\right)\lambda\rho-\kappa\xi}{1-\beta\rho+\kappa\xi} > 1-\rho+\frac{\kappa\xi\rho-\kappa\xi}{1-\beta\rho+\kappa\xi} = \frac{(1-\rho)(1-\beta\rho)}{1-\beta\rho+\kappa\xi} > 0.$$

Using $\varrho \equiv \frac{1}{\rho}-1$ and $r \equiv 1-\beta$, we can rewrite (9) as $\rho < \frac{\kappa\xi}{1-\beta+\kappa\xi}$. If $\rho < \frac{\kappa\xi}{1-\beta+\kappa\xi}$, then

$$\lambda < \frac{\alpha\left(1-\beta+\kappa\xi\right)}{s\kappa\left(1-\alpha\right)} < \frac{\alpha\xi}{s\left(1-\alpha\right)\rho}$$

Having $\lambda < \frac{\alpha\xi}{s(1-\alpha)\rho}$, we obtain

$$s\kappa\left(1/\alpha-1\right)\lambda\rho-\kappa\xi < 0$$

Hence

$$\frac{1-\rho}{1-\rho+\frac{s\kappa(1/\alpha-1)\lambda\rho-\kappa\xi}{1-\beta\rho+\kappa\xi}} > 1$$

□

A.5 Proof of Lemma 3

Proof. Recall that γ is

$$\gamma = \frac{\kappa\left(1/\alpha-1\right)\left(1-s\right)\left(1-\rho\right)}{1+\beta\rho^2+\left[s\kappa\left(1/\alpha-1\right)\lambda-\left(1+\kappa\xi+\beta\right)\right]\rho}$$

Because $\rho < 1$, γ is positive. An increase in λ increases the denominator on the right-hand side of the above equation and therefore decreases γ. Next, $\chi > 0$. It is given by

$$\chi = \frac{(1 + \kappa\xi - \rho\beta)\,\gamma - \kappa\,(1/\alpha - 1)\,(1 - s)}{s\kappa\,(1/\alpha - 1)}$$

Thus, a decrease in γ resulting from an increase in λ reduces χ. $\square$

A.6 Proof of Lemma 4

Proof. Recall that γ is

$$\gamma = \frac{\kappa\,(1/\alpha - 1)\,(1 - s)\,(1 - \rho)}{1 + \beta\rho^2 + \rho\,[s\kappa\,(1/\alpha - 1)\,\lambda - (1 + \kappa\xi + \beta)]}$$

Then, we have

$$\frac{\partial\gamma}{\partial\rho} = \frac{-\kappa\,(1/\alpha - 1)\,(1 - s)[-\beta(1 - \rho)^2 - \kappa\xi + s\kappa\,(1/\alpha - 1)\,\lambda]}{\triangle^2}$$

where $\triangle = 1 + \beta\rho^2 + \rho\,[s\kappa\,(1/\alpha - 1)\,\lambda - (1 + \kappa\xi + \beta)]$.
The root of the equation $-\beta(1 - \rho)^2 - \kappa\xi + s\kappa\,(1/\alpha - 1)\,\lambda = 0$ is $1 - (\frac{\kappa[s(1/\alpha-1)\lambda-\xi]}{\beta})^{\frac{1}{2}}$. Given the restriction imposed on λ that ensures a locally unique equilibrium, we can show that as long as $\beta > 0.5$, $1 - (\frac{\kappa[s(1/\alpha-1)\lambda-\xi]}{\beta})^{\frac{1}{2}} \in (0, 1)$. Thus, if $\rho < 1 - (\frac{\kappa[s(1/\alpha-1)\lambda-\xi]}{\beta})^{\frac{1}{2}}$, we have $\frac{\partial\gamma}{\partial\rho} > 0$. And if $\rho > 1 - (\frac{\kappa[s(1/\alpha-1)\lambda-\xi]}{\beta})^{\frac{1}{2}}$, $\frac{\partial\gamma}{\partial\rho} < 0$. $\square$

Appendix B Intuition for Lemma 4

The starting point is:

$$E_t\,(\hat{\pi}_{t+2}) + \beta^{-1}\,(\kappa\,[s\lambda\,(1 - \alpha) - \xi] - 1 - \beta)\,E_t\,(\hat{\pi}_{t+1}) + \beta^{-1}\hat{\pi}_t = -\kappa\,(1 - s)\,(1 - \alpha)\,E_t\,(\Delta\hat{g}_{t+1})$$

Using lag operator notation:

$$\left[1 + \beta^{-1}\,(\kappa\,[s\lambda\,(1 - \alpha) - \xi] - 1 - \beta)\,L + \beta^{-1}L^2\right] E_t\,(\hat{\pi}_{t+2}) = -\kappa\,(1 - s)\,(1 - \alpha)\,E_t\,(\Delta\hat{g}_{t+1})$$

Note that the appearence of ξ only works to make the monetary policy less "active". Other than this, it does not influence the dynamics of inflation. Therefore, define $\tilde{\lambda} = s\lambda(1-\alpha) - \xi$.

$$\left[1 + \beta^{-1}\left(\kappa\tilde{\lambda} - 1 - \beta\right)L + \beta^{-1}L^2\right] E_t\left(\hat{\pi}_{t+2}\right) = -\kappa(1-s)(1-\alpha)E_t\left(\Delta\hat{g}_{t+1}\right)$$

Next, factoring the lag polynomial gives us:

$$(1-\Lambda_1 L)(1-\Lambda_2 L)E_t\left(\hat{\pi}_{t+2}\right) = -\kappa(1-s)(1-\alpha)E_t\left(\Delta\hat{g}_{t+1}\right)$$

It is possible to prove both roots are unstable, therefore we solve them forward.

$$(-\Lambda_1 L)(-\Lambda_1 L)^{-1}(-\Lambda_2 L)(-\Lambda_2 L)^{-1}(1-\Lambda_1 L)(1-\Lambda_2 L)E_t\left(\hat{\pi}_{t+2}\right) = -\kappa(1-s)(1-\alpha)E_t\left(\Delta\hat{g}_{t+1}\right)$$

$$(\Lambda_1 L)(\Lambda_2 L)\left[1-(\Lambda_1 L)^{-1}\right]\left[1-(\Lambda_2 L)^{-1}\right]E_t\left(\hat{\pi}_{t+2}\right) = -\kappa(1-s)(1-\alpha)E_t\left(\Delta\hat{g}_{t+1}\right)$$

Rearranging,

$$\hat{\pi}_t = \frac{-\kappa(1-s)(1-\alpha)}{\Lambda_1\Lambda_2}\left[1-(\Lambda_1 L)^{-1}\right]E_t\left\{\Delta\hat{g}_{t+1} + (\Lambda_2)^{-1}\Delta\hat{g}_{t+2} + (\Lambda_2)^{-2}\Delta\hat{g}_{t+3} + ...\right\}$$

Taking this out even further,

$$\begin{aligned}\hat{\pi}_t \;=\; & \frac{-\kappa(1-s)(1-\alpha)}{\Lambda_1\Lambda_2}E_t\left\{\Delta\hat{g}_{t+1} + (\Lambda_2)^{-1}\Delta\hat{g}_{t+2} + (\Lambda_2)^{-2}\Delta\hat{g}_{t+3} + ...\right.\\ & (\Lambda_1)^{-1}\left[\Delta\hat{g}_{t+2} + (\Lambda_2)^{-1}\Delta\hat{g}_{t+3} + (\Lambda_2)^{-2}\Delta\hat{g}_{t+4} + ...\right] + ...\\ & \left.(\Lambda_2)^{-1}\left[\Delta\hat{g}_{t+3} + (\Lambda_2)^{-1}\Delta\hat{g}_{t+4} + (\Lambda_2)^{-2}\Delta\hat{g}_{t+5} + ...\right] + ...\right\}\end{aligned}$$

Simplyifying further

$$\begin{aligned}\hat{\pi}_t \;=\; & \frac{-\kappa(1-s)(1-\alpha)(\rho-1)}{\Lambda_1\Lambda_2}\left\{1 + \rho(\Lambda_2)^{-1} + \rho^2(\Lambda_2)^{-2} + ...\right.\\ & (\Lambda_1)^{-1}\left[\rho + (\Lambda_2)^{-1}\rho^2 + (\Lambda_2)^{-2}\rho^3 + ...\right] + ...\\ & \left.(\Lambda_2)^{-1}\left[\rho^2 + (\Lambda_2)^{-1}\rho^3 + (\Lambda_2)^{-2}\rho^4 + ...\right] + ...\right\}\hat{g}_t\end{aligned}$$

Doing it again,

$$\begin{aligned}\hat{\pi}_t &= \frac{-\kappa(1-s)(1-\alpha)(\rho-1)}{\Lambda_1\Lambda_2}\left\{1+\rho(\Lambda_2)^{-1}+\rho^2(\Lambda_2)^{-2}+...\right. \\ &\quad (\Lambda_1)^{-1}\left[\rho+(\Lambda_2)^{-1}\rho^2+(\Lambda_2)^{-2}\rho^3+...\right]+... \\ &\quad \left.(\Lambda_1)^{-2}\left[\rho^2+(\Lambda_2)^{-1}\rho^3+(\Lambda_2)^{-2}\rho^4+...\right]+...\right\}\hat{g}_t\end{aligned}$$

Getting there,

$$\hat{\pi}_t = \frac{-\kappa(1-s)(1-\alpha)(\rho-1)}{\Lambda_1\Lambda_2}\left\{\frac{1}{1-(\rho/\Lambda_2)}+\left(\frac{\rho}{\Lambda_1}\right)\left(\frac{1}{1-(\rho/\Lambda_2)}\right)+\left(\frac{\rho}{\Lambda_1}\right)^2\left(\frac{1}{1-(\rho/\Lambda_2)}\right)+...\right\}\hat{g}_t$$

$$\hat{\pi}_t = \frac{-\kappa(1-s)(1-\alpha)(\rho-1)}{\Lambda_1\Lambda_2}\left\{\frac{1}{[1-(\rho/\Lambda_2)][1-(\rho/\Lambda_1)]}\right\}\hat{g}_t$$

$$\hat{\pi}_t = -\kappa(1-s)(1-\alpha)(\rho-1)\left\{\frac{1}{[\Lambda_1-\rho][\Lambda_2-\rho]}\right\}\hat{g}_t$$

Appendix C Lemma 5

Let $\rho \in (0,1)$. Let $\underline{\rho}$ denote the smaller root of the equation $\beta s\kappa(1/\alpha-1)\lambda\rho^2 - 2\beta\kappa\xi\rho + \kappa\xi(1+\kappa\xi+\beta) - (1+\kappa\xi)s\kappa(1/\alpha-1)\lambda$. If the parameter set is configured such that $\underline{\rho} > 0$ (i.e. $\lambda < \frac{(1+\kappa\xi+\beta)\alpha\xi}{(1+\kappa\xi)s(1-\alpha)}$), then the impact response of consumption to a government spending shock is increasing in ρ if $\rho < \underline{\rho}$, and it is decreasing in ρ if the inequality is reversed.

Proof. The partial derivative of χ with respect to ρ is given by

$$\frac{\partial\chi}{\partial\rho} = \frac{1-s}{s}\left[\frac{\beta s\kappa(1/\alpha-1)\lambda\rho^2 - 2\beta\kappa\xi\rho + \kappa\xi(1+\kappa\xi+\beta) - (1+\kappa\xi)s\kappa(1/\alpha-1)\lambda}{\Omega^2(1-\beta\rho+\kappa\xi)^2(1-\rho)^2}\right]$$

where $\Omega = 1 + \frac{s\kappa(1/\alpha-1)\lambda\rho-\kappa\xi}{(1-\beta\rho+\kappa\xi)(1-\rho)}$.
Let $\underline{\rho}$ and $\overline{\rho}$ denote the two roots of the equation $\beta s\kappa(1/\alpha-1)\lambda\rho^2 - 2\beta\kappa\xi\rho + \kappa\xi(1+\kappa\xi+\beta) - (1+\kappa\xi)s\kappa(1/\alpha-1)\lambda = 0$. Hence, if $\rho < \underline{\rho}$ or $\rho > \overline{\rho}$, we have $\frac{\partial\chi}{\partial\rho} > 0$. And if $\underline{\rho} < \rho < \overline{\rho}$, $\frac{\partial\chi}{\partial\rho} < 0$.
We can show that $\underline{\rho} < 1$. Moreover, $\overline{\rho} > 1$ because of our restriction that guarantees a unique equilibrium.
If $\lambda < \frac{(1+\kappa\xi+\beta)\alpha\xi}{(1+\kappa\xi)s(1-\alpha)}$, then $\underline{\rho} > 0$. Therefore, $\frac{\partial\chi}{\partial\rho} > 0$ when $\rho < \underline{\rho}$, and $\frac{\partial\chi}{\partial\rho} < 0$ when $\rho > \underline{\rho}$. □

Establishing an intuition for Lemma 5 is challenging. If $\lambda > \frac{(1+\kappa\xi+\beta)\alpha\xi}{(1+\kappa\xi)s(1-\alpha)}$, then $\underline{\rho} < 0$.

The real interest rate channel with an active monetary policy is strong enough so that the size of the response of consumption on impact decreases in $\rho \in (0,1)$. Lemma 5 focuses on the case where the impact response of consumption exhibits a hump-shaped pattern. This pattern also results from the interactions among the negative wealth effect, the real interest rate channel and the real wage channel, as explained in Lemma 4. One thing to add is that as long as $\beta > 0.2$, the peak of the consumption response occurs at a smaller ρ than that of the inflation response does, i.e. $\underline{\rho} < 1 - (\frac{\kappa[s(1/\alpha-1)\lambda-\xi]}{\beta})^{\frac{1}{2}}$.

Mathematically, $\frac{\partial\chi}{\partial\rho}$ can be written as

$$\frac{\partial\chi}{\partial\rho} = \frac{\alpha(1+\kappa\xi-\beta\rho)}{\kappa s(1-\alpha)}\frac{\partial\gamma}{\partial\rho} - \frac{\alpha\beta\gamma}{\kappa s(1-\alpha)}$$

The first term is the influence from current inflation and it governs the effects of the real wage channel. This term varies with $\frac{\partial\gamma}{\partial\rho}$, the change in the response of contemporaneous inflation with respect to the persistency of the government spending shock. The second term is related to the size of the increase in the real interest rate because it reflects the response of expected inflation. A higher expected inflation caused by a more persistent government spending shock would result in a larger real interest rate provided that the monetary policy is active. Consequently, a larger real interest rate prevents private consumption from increasing. Because the negative sign of the second term, the peak of the consumption response occurs earlier than that of the inflation response.